

Recall: A rep'n of a gp G (f.t.e.) on a vector space V (over $\mathbb{C}$) is $\rho: G \rightarrow GL(V)$, a homomorphism

Similar def'n

A matrix rep'n of G is a homom. $\rho: G \rightarrow GL_n(\mathbb{C})$, some n .

Note: If V is a $\mathbb{C}$ -space, picking a basis for V identifies $GL(V)$ with $GL_n(\mathbb{C})$

Suppose $\rho: G \rightarrow GL(V)$
 $\rho': G \rightarrow GL(V')$

A linear map $T: V \rightarrow V'$ is said to be G -equivariant

if $\boxed{T \rho(g) = \rho'(g) T} \quad \forall g \in G.$

$$\begin{array}{ccc} V & \xrightarrow{T} & V' \\ \rho(g) \downarrow & \circlearrowleft & \downarrow \rho'(g) \\ V & \xrightarrow{T} & V' \end{array}$$

Def'n Two representations $\rho \in V$ and $\rho' \in V'$ are said to be equivalent or isomorphic if $\exists T: V \rightarrow V'$ which is G -equiv linear isom.

Last time: $D_n \underset{p}{G} \mathbb{C}^2$ and $D_n \underset{p'}{G} \mathbb{C}^2$
 $D_n = \langle x, y \mid x^n = 1, y^2 = 1, yxy^{-1} = x^{-1} \rangle$

$$\rho(x) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}, \quad \theta = \frac{2\pi}{n}$$

$$\rho(y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\rho'(x) = \begin{pmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{pmatrix}, \quad \rho'(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Claim: The reps ρ and ρ' are equivalent.

Define: $T: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ as

$$\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

Note: (1) T is a lin isom.

$$(2) \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \underbrace{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}}$$

$$(3) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

(2) + (3) $\Rightarrow T$ is equiv

② + ③ $\Rightarrow T$ is equiv

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{T} & \mathbb{C}^2 \\ p' \downarrow & & \downarrow p \\ \mathbb{C}^2 & \xrightarrow{T} & \mathbb{C}^2 \end{array}$$

Problem: Classify reps of G up to isomorphism

Lemma: Two matrix reps $\rho_1, \rho_2: G \rightarrow GL_n(\mathbb{C})$

define equiv reps $\Leftrightarrow \exists T \in GL_n(\mathbb{C})$ s.t.

$$\rho_1(g) = T \rho_2(g) T^{-1} \quad \forall g \in G.$$

In other words, $\Leftrightarrow \rho_1$ and ρ_2 are conjugate homoms.

$$G \rightarrow GL_n(\mathbb{C}).$$

Example: S_n on $\mathbb{C}^n$ by permutations

$$W = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} : \sum_{j=1}^n a_j = 0 \right\} \subseteq \mathbb{C}^n$$

saw before: S_n on W , i.e. if $p \in S_n$ and $w \in W$ then $p \cdot w \in W$.

Defⁿ Suppose $\rho \in V$ is a repⁿ, and $W \subseteq V$ is a subspace.

Then W is said to be a subrep or an invariant subspace.

Then W is said to be a subspace or an invariant subspace if $g \cdot w \in W \quad \forall g \in G \text{ and } w \in W$.

Another example: $G = S_n$, $V = \mathbb{C}^n$, $G \subset V$ by permutations.

Saw above: $W \subseteq V$ is invt.

Another subspace: Define $W' = \mathbb{C} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

Then W' is another invt subspace

Aside: $\oplus$ of spaces, § 3.6 of textbook

Suppose V_1, V_2 are $\mathbb{C}$ spaces. The direct sum of V_1 & V_2 is

$$V_1 \oplus V_2 = \{ (v_1, v_2) : v_1 \in V_1, \text{ and } v_2 \in V_2 \}$$

$V_1 \oplus V_2$ is a space with

- $\lambda(v_1, v_2) = (\lambda v_1, \lambda v_2)$ if $\lambda \in \mathbb{C}$, $v_1 \in V_1, v_2 \in V_2$
- $(v_1, v_2) + (v'_1, v'_2) = (v_1 + v'_1, v_2 + v'_2)$.

Similarly: $V_1 \oplus V_2 \oplus \dots \oplus V_n = \{ (v_1, v_2, \dots, v_n) : v_j \in V_j \}$

w/ addition & scalar mult defined componentwise

Lemma 1: $V_1, V_2 \subseteq V$ subspaces with

- $V_1 \cap V_2 = \{0\}$

- $V_1 \cap V_2 = \{0\}$
- $V_1 + V_2 = V$, i.e. every $v \in V$ is expressible as
 $v = v_1 + v_2 \quad \wedge \quad v_1 \in V_1 + v_2 \in V_2$

Then: $V_1 \oplus V_2 \cong V$

Pf: Define $\varphi: V_1 \oplus V_2 \rightarrow V$ as
 $\varphi(v_1, v_2) = v_1 + v_2$

Assumptions: $\Rightarrow \varphi$ is surj linear map $\Rightarrow \varphi$ is lin isom. $\square$
 • $\ker \varphi = 0$

Suppose $G \subseteq V_1$, $G \subseteq V_2$. Then $G \subseteq V_1 \oplus V_2$ as

$$g \cdot (v_1, v_2) = (\rho_1(g)v_1, \rho_2(g)v_2).$$

This is an action of G on $V_1 \oplus V_2$ by linear operators
 and thus defines a rep'n $\rho: G \rightarrow GL(V_1 \oplus V_2)$.

This is called the direct sum of the reps ρ_1 & ρ_2 .

If you pick a basis for V_1 and a basis for V_2

$$\leadsto \rho_1: G \rightarrow GL_{\dim(V_1)}(\mathbb{C})$$

$$\rho_2: G \rightarrow GL_{\dim(V_2)}(\mathbb{C})$$

$$\Rightarrow \rho: G \rightarrow GL_{\dim(V_1) + \dim(V_2)}(\mathbb{C}) \quad \text{as}$$

/ n (c) / $\rightarrow$ 1

1

$$\dim(V_1) + \dim(V_2)$$

$$\rho(g) = \left(\begin{array}{c|c} \rho_1(g) & 0 \\ \hline 0 & \rho_2(g) \end{array} \right).$$

Example: $G = S_n$, $V = \mathbb{C}^n$, $\rho: S_n \rightarrow GL_n(\mathbb{C})$

$$V_1 = \mathbb{C} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad V_2 = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} : \sum_{j=1}^n a_j = 0 \right\}$$

- Then V_1, V_2 are invt subspaces of V
- $V = V_1 \oplus V_2$
- $S_n \curvearrowright V_1$ is called ρ_1 ; ρ_1 is the trivial repn
- $S_n \curvearrowright V_2$ is called ρ_2
- $\rho = \rho_1 \oplus \rho_2$

Example $G = \mathbb{Z}/3\mathbb{Z}$, $V = \mathbb{C}^3$

with $\rho: G \rightarrow GL(V) = GL_3(\mathbb{C})$

$$\text{as } \rho(1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho(2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Set $\omega = e^{2\pi i/3}$.

$$\text{Define } b_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix}$$

Then $\rho(1)b_0 = b_0$, $\rho(1)b_1 = \omega b_1$, $\rho(1)b_2 = \omega^2 b_2$

$\Rightarrow V_0 = \mathbb{C}b_0$, $V_1 = \mathbb{C}b_1$, $V_2 = \mathbb{C}b_2$ then

$$V = V_0 \oplus V_1 \oplus V_2$$

and this is a direct sum of reps.

Defⁿ A repⁿ V of a gr^p G is said to be irreducible if
 $W \subseteq V$ invariant $\Rightarrow W = \{0\}$ or $W = V$.

In other words, there are no nontrivial invariant subspaces.

Thm (Maschke's Thm) Suppose G a fte gr, V a fte $\mathbb{C}$ -rep of G . Then V is a direct sum of irreducible reps.

So, the irred. reps of a gr G are the "atoms" of repⁿ theory
